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Non-response and Estimation of Ratio of Two Population Means using Auxiliary Information

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SUMMARY

A class of estimators for ratio of two population mean of study variables using the knowledge of population mean of an auxiliary variable is proposed, its bias and mean square error are found. A sub-class of optimum estimators in the sense of having minimum mean square error is found and enhancing the practical utility, a sub-class of estimators depending on estimated optimum value based on sample observations is also investigated in the presence of non-response.

Keywords: A class of estimators, Auxiliary information, Non-response, Bias, Mean square error and efficiency.

1. INTRODUCTION

Let y_1 and y_2 be the two study variables of interest with their population means $\bar{Y}_1$ and $\bar{Y}_2$ respectively and x be the auxiliary variable with known population mean $\bar{X}$. For the case of non-response in sample survey, the procedure of sub-sampling the non-respondents was suggested by Hansen and Hurwitz (1946). Let n be the size of the simple random sample without replacement drawn from the population of size N where n_1 of selected n units respond and n_2 sample units do not respond.

From the non-response units, $r\left(=\frac{n_2}{k}, k>1\right)$ units are selected by making extra efforts and thus giving $n_1 + r$ observations on the study variables y_1 and y_2 in place of n. For $\bar{y}_1$ and $\bar{y}_2$ being the sample means based on n_1 units and $\bar{y}'_{1(2)}$ and $\bar{y}'_{2(2)}$ being the sample means based on r units, Hansen and Hurwitz (1946) using $n_1 + r$ observations on the y_i character gave the unbiased estimator of population mean given by $\bar{y}_i^*$ as

$$\bar{y}_i^* = \frac{n_1}{n}\bar{y}_i + \frac{n_2}{n}\bar{y}'_{i(2)} \tag{1.1}$$

which has its variance to be

$$Var(\bar{y}_i^*) = \frac{(1-f)}{n}S_{y_i}^2 + \frac{W_2(k-1)}{n}S_{y_{i(2)}}^2 \tag{1.2}$$

where $f = \frac{n}{N}$, $W_i = \frac{N_i}{N}$, (i = 1, 2) and $S_{y_i}^2$ and $S_{y_{i(2)}}^2$, (i = 1, 2) are the variances for the whole population and for the non-response group of the population respectively.

With $\bar{X}$ being known and incomplete information on y_1 and y_2, Hansen and Hurwitz (1946) conventional ratio estimator becomes

$$\hat{R}_0^* = \frac{\bar{y}_1^*}{\bar{y}_2^*}\frac{\bar{X}}{\bar{x}^*} \tag{1.3}$$

where $\bar{x}^* = \frac{n_1}{n}\bar{x}_1 + \frac{n_2}{n}\bar{x}'_{(2)}$ with $\bar{x}_1$ and $\bar{x}'_{(2)}$ being the sample means based on n_1 and r observations respectively. Further, when $\bar{X}$ known and incomplete

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information on y_1 and y_2, Khare and Srivastava (1997) transformed ratio type estimator becomes

$$\hat{R}_1^* = \frac{\bar{y}_1^*(\bar{X}+L)}{\bar{y}_2^*(\bar{x}^*+L)} \tag{1.4}$$

where L is a positive constant.

Also, some other ratio type estimators may be given as

$$\hat{R}_2^* = \frac{\bar{y}_1^*(\bar{x}^*+L)}{\bar{y}_2^*(\bar{X}+L)} \tag{1.5}$$

$$\hat{R}_3^* = \frac{\bar{y}_1^*}{\bar{y}_2^*}\left(\frac{\bar{x}^*}{\bar{X}}\right)^{\alpha} \tag{1.6}$$

$$\hat{R}_4^* = \left(\frac{\bar{y}_1^*}{\bar{y}_2^*}\right)e^{\alpha\left(\frac{\bar{x}^*}{\bar{X}}-1\right)} \tag{1.7}$$

$$\hat{R}_5^* = \frac{\bar{y}_1^*}{\bar{y}_2^*}\left\{1+\alpha\frac{(\bar{x}^*-\bar{X})}{\bar{X}}\right\} \tag{1.8}$$

$$\hat{R}_6^* = \left(\frac{\bar{y}_1^*}{\bar{y}_2^*}\right)+\alpha(\bar{x}^*-\bar{X}) \tag{1.9}$$

$$\hat{R}_7^* = \left(\frac{\bar{y}_1^*}{\bar{y}_2^*}\right)+\left\{\left(\frac{\bar{x}^*}{\bar{X}}\right)^{\alpha}-1\right\} \tag{1.10}$$

$$\hat{R}_8^* = \frac{\{\bar{y}_1^*+\alpha(\bar{x}^*-\bar{X})\}}{\bar{y}_2^*} \tag{1.11}$$

Seeing the forms of the estimators from (1.3) to (1.11), we define a generalized class of estimators as a function $\hat{R}_g^* = g(\bar{y}_1^*, \bar{y}_2^*, \bar{x}^*)$ satisfying the validity conditions of Taylor's series expansion such that

$$\left.\begin{array}{l} (i)\ g(\bar{Y}_1, \bar{Y}_2, \bar{X}) = R \\ (ii)\ g_1 = \left.\dfrac{\partial g(\bar{y}_1^*, \bar{y}_2^*, \bar{x}^*)}{\partial \bar{y}_1^*}\right]_{P=(\bar{Y}_1,\bar{Y}_2,\bar{X})} = \dfrac{1}{\bar{Y}_2} \\ (iii)\ g_2 = \left.\dfrac{\partial g(\bar{y}_1^*, \bar{y}_2^*, \bar{x}^*)}{\partial \bar{y}_2^*}\right]_{P=(\bar{Y}_1,\bar{Y}_2,\bar{X})} = -\dfrac{\bar{Y}_1}{\bar{Y}_2^2} \\ (iv)\ g_0 = \left.\dfrac{\partial g(\bar{y}_1^*, \bar{y}_2^*, \bar{x}^*)}{\partial \bar{x}^*}\right]_{P=(\bar{Y}_1,\bar{Y}_2,\bar{X})} \end{array}\right\} \tag{1.12}$$

It may be mentioned here that all estimators listed from (1.3) to (1.11) belong to the generalized class $\hat{R}_g^*$ of estimators; hence, their results may be easily obtained as the special cases directly from the generalized class $\hat{R}_g^*$ of estimators.

2. BIAS AND MEAN SQUARE ERROR

Let us define,

$e_1^* = (\bar{y}_1^* - \bar{Y}_1)$, $e_2^* = (\bar{y}_2^* - \bar{Y}_2)$, $e_0^* = (\bar{x}^* - \bar{X})$ so that

$E(e_1^*) = E(e_2^*) = E(e_0^*) = 0$ and

$$E(e_1^{*2}) = \frac{(N-n)}{Nn}S_{y_1}^2 + \frac{(k-1)N_2}{Nn}S_{y_1(2)}^2,$$

$$E(e_2^{*2}) = \frac{(N-n)}{Nn}S_{y_2}^2 + \frac{(k-1)N_2}{Nn}S_{y_2(2)}^2,$$

$$E(e_0^{*2}) = \frac{(N-n)}{Nn}S_{x}^2 + \frac{(k-1)N_2}{Nn}S_{x(2)}^2,$$

$$E(e_0^*e_1^*) = \frac{(N-n)}{Nn}S_{y_1x} + \frac{(k-1)N_2}{Nn}S_{y_1x(2)},$$

$$E(e_0^*e_2^*) = \frac{(N-n)}{Nn}S_{y_2x} + \frac{(k-1)N_2}{Nn}S_{y_2x(2)},$$

$$E(e_1^*e_2^*) = \frac{(N-n)}{Nn}S_{y_1y_2} + \frac{(k-1)N_2}{Nn}S_{y_1y_2(2)}$$

where S_x^2 and $S_{x(2)}^2$ are the variances of x for the whole population and for the non-response group of the population respectively, $S_{y_ix}^2$ and $S_{y_ix(2)}^2$, (i = 1, 2) are respectively the covariances between y_i, (i = 1, 2) and x for the whole population and for the non-response group of the population, and $S_{y_1y_2}$ and $S_{y_1y_2(2)}$ are respectively the covariances between y_1 and y_2 for the whole population and for the non-response group of the population.

Expanding $\hat{R}_g^* = g(\bar{y}_1^*, \bar{y}_2^*, \bar{x}^*)$ about the point $P = (\bar{Y}_1, \bar{Y}_2, \bar{X})$ in third order Taylor's series, we have

$$\begin{aligned}\hat{R}_g^* &= g(\bar{Y}_1,\bar{Y}_2,\bar{X}) + (\bar{y}_1^*-\bar{Y}_1)g_1 + (\bar{y}_2^*-\bar{Y}_2)g_2 + (\bar{x}^*-\bar{X})g_0 \\ &+\frac{1}{2!}\{(\bar{y}_1^*-\bar{Y}_1)^2 g_{11} + (\bar{y}_2^*-\bar{Y}_2)^2 g_{22} + (\bar{x}^*-\bar{X})^2 g_{00} \\ &+2(\bar{y}_1^*-\bar{Y}_1)(\bar{x}^*-\bar{X})g_{10} + 2(\bar{y}_2^*-\bar{Y}_2)(\bar{x}^*-\bar{X})g_{20} \\ &+2(\bar{y}_1^*-\bar{Y}_1)(\bar{y}_2^*-\bar{Y}_2)g_{12}\}\end{aligned}$$

$$+\frac{1}{3!}(\bar{y}_1^* - \bar{Y}_1)\frac{\partial}{\partial \bar{y}_1^*} + (\bar{y}_2^* - \bar{Y}_2)\frac{\partial}{\partial \bar{y}_2^*} + (\bar{x}^* - \bar{X})\frac{\partial}{\partial \bar{x}^*}\Bigg\}^3$$

$$\times g(\bar{y}_{1*}^*, \bar{y}_{2*}^*, \bar{x}_*^*) \qquad (2.1)$$

where $g_1 = \dfrac{\partial g(\bar{y}_1^*, \bar{y}_2^*, \bar{x}^*)}{\partial \bar{y}_1^*}\Bigg]_{P=(\bar{Y}_1, \bar{Y}_2, \bar{X})} = \dfrac{1}{\bar{Y}_2}$,

$$g_2 = \frac{\partial g(\bar{y}_1^*, \bar{y}_2^*, \bar{x}^*)}{\partial \bar{y}_2^*}\Bigg]_{P=(\bar{Y}_1, \bar{Y}_2, \bar{X})} = -\frac{\bar{Y}_1}{\bar{Y}_2^2},$$

$$g_0 = \frac{\partial g(\bar{y}_1^*, \bar{y}_2^*, \bar{x}^*)}{\partial \bar{x}^*}\Bigg]_{P=(\bar{Y}_1, \bar{Y}_2, \bar{X})},$$

$$g_{11} = \frac{\partial^2 g(\bar{y}_1^*, \bar{y}_2^*, \bar{x}^*)}{\partial \bar{y}_1^{*2}}\Bigg]_{P=(\bar{Y}_1, \bar{Y}_2, \bar{X})} = 0,$$

$$g_{22} = \frac{\partial^2 g(\bar{y}_1^*, \bar{y}_2^*, \bar{x}^*)}{\partial \bar{y}_2^{*2}}\Bigg]_{P=(\bar{Y}_1, \bar{Y}_2, \bar{X})} = \frac{2\bar{Y}_1}{\bar{Y}_2^3},$$

$$g_{00} = \frac{\partial^2 g(\bar{y}_1^*, \bar{y}_2^*, \bar{x}^*)}{\partial \bar{x}^{*2}}\Bigg]_{P=(\bar{Y}_1, \bar{Y}_2, \bar{X})},$$

$$g_{10} = \frac{\partial^2 g(\bar{y}_1^*, \bar{y}_2^*, \bar{x}^*)}{\partial \bar{y}_1^* \partial \bar{x}^*}\Bigg]_{P=(\bar{Y}_1, \bar{Y}_2, \bar{X})},$$

$$g_{20} = \frac{\partial^2 g(\bar{y}_1^*, \bar{y}_2^*, \bar{x}^*)}{\partial \bar{y}_2^* \partial \bar{x}^*}\Bigg]_{P=(\bar{Y}_1, \bar{Y}_2, \bar{X})},$$

$$g_{12} = \frac{\partial^2 g(\bar{y}_1^*, \bar{y}_2^*, \bar{x}^*)}{\partial \bar{y}_1^* \partial \bar{y}_2^*}\Bigg]_{P=(\bar{Y}_1, \bar{Y}_2, \bar{X})} = -\frac{1}{\bar{Y}_2^2}$$

and $\bar{y}_{1*}^* = \bar{Y}_1 + \theta(\bar{y}_1^* - \bar{Y}_1), \bar{y}_{2*}^* = \bar{Y}_2 + \theta(\bar{y}_2^* - \bar{Y}_2)$,

$\bar{x}_*^* = \bar{X} + \theta(\bar{x}^* - \bar{X})$ for $0 < \theta < 1$.

On substituting $g(\bar{Y}_1, \bar{Y}_2, \bar{X}) = R$ and the values of the derivatives in (2.1), we have

$$\hat{R}_g^* = R + (\bar{y}_1^* - \bar{Y}_1)\left(\frac{1}{\bar{Y}_2}\right) + (\bar{y}_2^* - \bar{Y}_2)\left(-\frac{\bar{Y}_1}{\bar{Y}_2^2}\right)$$

$$+(\bar{x}^* - \bar{X})g_0 + \frac{1}{2!}\left\{(\bar{y}_1^* - \bar{Y}_1)^2.0 + (\bar{y}_2^* - \bar{Y}_2)^2\left(2\frac{\bar{Y}_1}{\bar{Y}_2^3}\right)\right.$$

$$+(\bar{x}^* - \bar{X})^2 g_{00} + 2(\bar{y}_1^* - \bar{Y}_1)(\bar{x}^* - \bar{X})g_{10}$$

$$+2(\bar{y}_2^* - \bar{Y}_2)(\bar{x}^* - \bar{X})g_{20}$$

$$\left.+2(\bar{y}_1^* - \bar{Y}_1)(\bar{y}_2^* - \bar{Y}_2)\left(-\frac{1}{\bar{Y}_2^2}\right)\right\}$$

$$+\frac{1}{3!}\left\{(\bar{y}_1^* - \bar{Y}_1)\frac{\partial}{\partial \bar{y}_1^*} + (\bar{y}_2^* - \bar{Y}_2)\frac{\partial}{\partial \bar{y}_2^*} + (\bar{x}^* - \bar{X})\frac{\partial}{\partial \bar{x}^*}\right\}^3$$

$$\times g(\bar{y}_{1*}^*, \bar{y}_{2*}^*, \bar{x}_*^*)$$

or $\hat{R}_g^* - R = \dfrac{e_1^*}{\bar{Y}_2} - \dfrac{e_2^* R}{\bar{Y}_2} + e_0^* g_0$

$$+\frac{1}{2!}\left\{\frac{2Re_2^{*2}}{\bar{Y}_2^2} + e_0^{*2} g_{00} + 2e_1^* e_0^* g_{10}\right.$$

$$\left.+2e_2^* e_0^* g_{20} - \frac{2e_1^* e_2^*}{Y_2^2}\right\}$$

$$+\frac{1}{3!}\left\{(\bar{y}_1^* - \bar{Y}_1)\frac{\partial}{\partial \bar{y}_1^*} + (\bar{y}_2^* - \bar{Y}_2)\frac{\partial}{\partial \bar{y}_2^*}\right.$$

$$\left.+(\bar{x}^* - \bar{X})\frac{\partial}{\partial \bar{x}^*}\right\}^3 g(\bar{y}_{1*}^*, \bar{y}_{2*}^*, \bar{x}_*^*) \qquad (2.2)$$

Taking expectation on both sides of (2.2) and ignoring terms in e_i's (i = 0, 1, 2) greater than two, the bias of $\hat{R}_g^*$ to the first degree of approximation {that is, up to terms of order $O\left(\frac{1}{n}\right)$} is

$$E(\hat{R}_g^*) - R = \frac{E(e_1^*)}{\bar{Y}_2} - \frac{RE(e_2^*)}{\bar{Y}_2} + E(e_0^*)g_0$$

$$+\frac{1}{2!}\left\{\frac{2RE(e_2^{*2})}{\bar{Y}_2^2} + E(e_0^{*2})g_{00} + 2E(e_1^* e_0^*)g_{10}\right.$$

$$\left.+2E\left(e_2^* e_0^*\right)g_{20} - \frac{2E\left(e_1^* e_2^*\right)}{\bar{Y}_2^2}\right\}$$

$$= \frac{R}{\bar{Y}_2^2}\left\{\frac{(1-f)}{n}S_{y_2}^2 + \frac{(k-1)W_2}{n}S_{y_2(2)}^2\right\}$$

$$+\frac{1}{2}\left\{\frac{(1-f)}{n}S_x^2 + \frac{(k-1)W_2}{n}S_{x(2)}^2\right\}g_{00}$$

$$+\left\{\frac{(1-f)}{n}S_{y_1x} + \frac{(k-1)W_2}{n}S_{y_1x(2)}\right\}g_{10}$$

$$+\left\{\frac{(1-f)}{n}S_{y_2x} + \frac{(k-1)W_2}{n}S_{y_2x(2)}\right\}g_{20}$$

$$-\frac{1}{\bar{Y}_2^2}\left\{\frac{(1-f)}{n}S_{y_1y_2} + \frac{(k-1)W_2}{n}S_{y_1y_2(2)}\right\}$$

or $Bias(\hat{R}_g^*)$

$$= \frac{(1-f)}{n}\left\{\frac{RS_{y_2}^2}{\bar{Y}_2^2} + \frac{S_x^2}{2}g_{00} + S_{y_1x}g_{10} + S_{y_2x}g_{20} - \frac{S_{y_1y_2}}{\bar{Y}_2^2}\right\}$$

$$+\frac{(k-1)N_2}{Nn}\left\{\frac{RS_{y_{2(2)}}^2}{\bar{Y}_2^2} + \frac{S_{x(2)}^2}{2}g_{00} + S_{y_1x_{(2)}}g_{10} + S_{y_2x_{(2)}}g_{20} - \frac{S_{y_1y_{2(2)}}}{\bar{Y}_2^2}\right\} \quad (2.3)$$

which shows that bias of $\hat{R}_g^*$ is of order $O\left(\frac{1}{n}\right)$; hence, for sufficiently large value of n, the bias is negligible.

Squaring both the sides of (2.2), taking expectation and ignoring terms in e_i's (i = 0, 1, 2) greater than two, the mean square error of $\hat{R}_g^*$ to the first degree of approximation {that is, up to terms of order $O\left(\frac{1}{n}\right)$} is

$$E(\hat{R}_g^* - R)^2 = E\left[\frac{e_1^*}{\bar{Y}_2} - \frac{Re_2^*}{\bar{Y}_2} + e_0^* g_0\right]^2$$

$$= \frac{E(e_1^{*2})}{\bar{Y}_2^2} + \frac{R^2E(e_2^{*2})}{\bar{Y}_2^2} - \frac{2RE(e_1^*e_2^*)}{\bar{Y}_2^2} + E(e_0^{*2})g_0^2 + \frac{2E(e_1^*e_0^*)}{\bar{Y}_2}g_0 - \frac{2RE(e_2^*e_0^*)}{\bar{Y}_2}g_0$$

$$= \frac{1}{\bar{Y}_2^2}\left\{\frac{(1-f)}{n}S_{y_1}^2 + \frac{(k-1)W_2}{n}S_{y_{1(2)}}^2\right\} + \frac{R^2}{\bar{Y}_2^2}\left\{\frac{(1-f)}{n}S_{y_2}^2 + \frac{(k-1)W_2}{n}S_{y_{2(2)}}^2\right\}$$

$$-\frac{2R}{\bar{Y}_2^2}\left\{\frac{(1-f)}{n}S_{y_1y_2} + \frac{(k-1)W_2}{n}S_{y_1y_{2(2)}}\right\} + \left[\frac{(1-f)}{n}S_x^2 + \frac{(k-1)W_2}{n}S_{x(2)}^2\right]g_0^2$$

$$+\frac{2}{\bar{Y}_2}\left\{\frac{(1-f)}{n}S_{y_1x} + \frac{(k-1)W_2}{n}S_{y_1x_{(2)}}\right\}g_0 - \frac{2R}{\bar{Y}_2}\left\{\frac{(1-f)}{n}S_{y_2x} + \frac{(k-1)W_2}{n}S_{y_2x_{(2)}}\right\}g_0$$

$$MSE(\hat{R}_g^*) = MSE(\hat{R}^*) + \frac{(1-f)}{n}\left[S_x^2g_0^2 + \frac{2}{\bar{Y}_2}S_{y_1x}g_0 - \frac{2R}{\bar{Y}_2}S_{y_2x}g_0\right] + \frac{(k-1)W_2}{n}\left[S_{x(2)}^2g_0^2 + \frac{2}{\bar{Y}_2}S_{y_1x_{(2)}}g_0 - \frac{2R}{\bar{Y}_2}S_{y_2x_{(2)}}g_0\right] \quad (2.4)$$

where $\hat{R}^* = \frac{\bar{y}_1^*}{\bar{y}_2^*}$.

The optimum value of minimizing the mean square error of $\hat{R}_g^*$ is

$$g_0^* = \frac{[A]}{\left[\frac{(1-f)}{n}S_x^2 + \frac{(k-1)W_2}{n}S_{x(2)}^2\right]} = G \quad (2.5)$$

where $A = -\frac{1}{\bar{Y}_2}\left\{\frac{(1-f)}{n}\{S_{y_1x} - RS_{y_2x}\} + \frac{(k-1)W_2}{n}\{S_{y_1x_{(2)}} - RS_{y_2x_{(2)}}\}\right\}$

$f = \frac{n}{N}$ and the minimum mean square error of and $\hat{R}_g^*$ is

$$MSE(\hat{R}_g^*)_{\min} = MSE(\hat{R}^*) + \frac{\left[-\frac{1}{\bar{Y}_2}\left\{\frac{(1-f)}{n}\{S_{y_1x} - RS_{y_2x}\} + \frac{(k-1)W_2}{n}\{S_{y_1x_{(2)}} - RS_{y_2x_{(2)}}\}\right\}\right]^2}{\left[\frac{(1-f)}{n}S_x^2 + \frac{(k-1)W_2}{n}S_{x(2)}^2\right]}$$

$$-2\frac{\left[\frac{1}{\bar{Y}_2}\left\{\frac{(1-f)}{n}\{S_{y_1x} - RS_{y_2x}\} + \frac{(k-1)W_2}{n}\{S_{y_1x_{(2)}} - RS_{y_2x_{(2)}}\}\right\}\right]^2}{\left[\frac{(1-f)}{n}S_x^2 + \frac{(k-1)W_2}{n}S_{x(2)}^2\right]}$$

$$= MSE(\hat{R}^*) - \frac{\left[\frac{1}{\bar{Y}_2}\left\{\frac{(1-f)}{n}\{S_{y_1x} - RS_{y_2x}\} + \frac{(k-1)W_2}{n}\{S_{y_1x_{(2)}} - RS_{y_2x_{(2)}}\}\right\}\right]^2}{\left[\frac{(1-f)}{n}S_x^2 + \frac{(k-1)W_2}{n}S_{x(2)}^2\right]} \quad (2.6)$$

The optimum value of g_0^* in (2.5) contains some unknown parameters and lacks its practical utility to attain the minimum mean square error of $\hat{R}_g^*$ in (2.6); hence, the alternative is to replace g_0^* by its consistent estimated optimum value $\hat{g}_0^*$ based on sample observations.

From Rao (1990), the unbiased estimators of $S_{y_ix}, S_x^2, S_{y_ix(2)}$ and $S_{x(2)}^2$ are respectively given by

$$\hat{S}_{y_ix} = \frac{1}{(n-1)}\left[(n_1-1)s_{y_ix_1} + \left\{\frac{n_2}{r}(r-1) + \frac{(k-1)W_2}{n}\right\}s_{y_ix(2)} + nW_1W_2(\bar{x}_1 - \bar{x}'_{(2)})(\bar{y}_i - \bar{y}'_{i(2)})\right];\ (i = 1, 2)$$

$$\hat{S}_x^2 = \frac{1}{(n-1)}\left[(n_1-1)s_{x_1}^2 + \left\{\frac{n_2}{r}(r-1) + \frac{(k-1)W_2}{n}\right\}s_{x(2)}^2 + nW_1W_2\left(\bar{x}_1 - \bar{x}'_{(2)}\right)^2\right]$$

$$s_{y_ix_1} = \frac{1}{(n_1-1)}\sum_{j=1}^{n_1}(x_j - \bar{x}_1)(y_{ij} - \bar{y}_i)$$

$$\hat{S}_{y_ix(2)} = s_{y_ix(2)} = \frac{1}{(r-1)}\sum_{j=1}^{r}(x_{j2} - \bar{x}'_{(2)})(y_{ij2} - \bar{y}'_{i(2)})$$

$$s_{x_1}^2 = \frac{1}{(n_1-1)}\sum_{j=1}^{n_1}(x_j - \bar{x}_1)^2$$

$$\hat{S}_{x(2)}^2 = s_{x(2)}^2 = \frac{1}{(r-1)}\sum_{j=1}^{r}(x_{j2} - \bar{x}'_{(2)})^2$$

which when substituted in g_0^* in (2.5) gives the estimated optimum value

$$\hat{g}_0^* = \frac{-\frac{1}{\bar{y}_2^*}\left\{\frac{(1-f)}{n}\{\hat{S}_{y_1x} - \hat{R}^*\hat{S}_{y_2x}\} + \frac{(k-1)W_2}{n}\{\hat{S}_{y_1x_{(2)}} - \hat{R}^*\hat{S}_{y_2x_{(2)}}\}\right\}}{\left[\frac{(1-f)}{n}\hat{S}_x^2 + \frac{(k-1)W_2}{n}\hat{S}_{x_{(2)}}^2\right]}$$

$$= \hat{G} \tag{2.7}$$

where x_{j2} and y_{ij2} denote the observations on the j^{th} unit of the sub-sampling units selected from n_2 non-respondent units.

To attain the minimum mean square error of $\hat{R}_g^*$ in (2.6), the function $\hat{R}_g^* = g(\bar{y}_1^*, \bar{y}_2^*, \bar{x}^*)$ should not only involve $(\bar{y}_1^*, \bar{y}_2^*, \bar{x}^*)$ satisfying (i) to (iv) in (1.12) but also $g_0^* = G$ in (2.5), but $g_0^* = G$ is unknown depending on parameters; hence, we should have the estimator as a function $\hat{R}_{ge}^* = g(\bar{y}_1^*, \bar{y}_2^*, \bar{x}^*, \hat{G})$ depending on estimated optimum $\hat{g}_0^* = \hat{G}$ in (2.7) and find the conditions which make the estimator $\hat{R}_{ge}^* = g(\bar{y}_1^*, \bar{y}_2^*, \bar{x}^*, \hat{G})$ having its mean square error to be equal to the minimum mean square error of $\hat{R}_g^*$ in (2.6).

Now expanding $\hat{R}_{ge}^* = g(\bar{y}_1^*, \bar{y}_2^*, \bar{x}^*, \hat{G})$ about the point $T = (\bar{Y}_1, \bar{Y}_2, \bar{X}, G)$ in Taylor's series, we have

$$\begin{aligned}\hat{R}_{ge}^* &= g(\bar{Y}_1, \bar{Y}_2, \bar{X}, G) + (\bar{y}_1^* - \bar{Y}_1)g_{1e} + (\bar{y}_2^* - \bar{Y}_2)g_{2e} \\ &+ \left(\bar{x}^* - \bar{X}\right)g_{0e} + (\hat{G} - G)g_{3e} + \frac{1}{2!}\{(\bar{y}_1^* - \bar{Y}_1)^2 g_{11e} \\ &+ (\bar{y}_2^* - \bar{Y}_2)^2 g_{22e} + (\bar{x}^* - \bar{X})^2 g_{00e} + (\hat{G} - G)^2 g_{33e} \\ &+ 2(\bar{y}_1^* - \bar{Y}_1)(\bar{y}_2^* - \bar{Y}_2)g_{12e} + 2\left(\bar{y}_1^* - \bar{Y}_1\right)\left(\bar{x}^* - \bar{X}\right)g_{10e} \\ &+ 2(\bar{y}_2^* - \bar{Y}_2)(\bar{x}^* - \bar{X})g_{20e} + 2(\bar{y}_1^* - \bar{Y}_1)(\hat{G} - G)g_{13e} \\ &+ 2(\bar{y}_2^* - \bar{Y}_2)(\hat{G} - G)g_{23e} + 2(\bar{x}^* - \bar{X})(\hat{G} - G)g_{03e}\} \\ &+ \ldots\end{aligned} \tag{2.8}$$

Similar to $\hat{R}_g^*$ from (i) to (iii) and (2.5), substituting in (2.8), we have

$$\begin{aligned}\hat{R}_{ge}^* &= R + (\bar{y}_1^* - \bar{Y}_1)\left(\frac{1}{\bar{Y}_2}\right) + (\bar{y}_2^* - \bar{Y}_2)\left(\frac{-\bar{Y}_1}{\bar{Y}_2^2}\right) + (\bar{x}^* - \bar{X})G \\ &+ (\hat{G} - G)g_{3e} + \frac{1}{2!}\{(\bar{y}_1^* - \bar{Y}_1)^2(0) + (\bar{y}_2^* - \bar{Y}_2)^2\left(\frac{2\bar{Y}_1}{\bar{Y}_2^3}\right) \\ &+ (\bar{x}^* - \bar{X})^2 g_{00e} + (\hat{G} - G)^2 g_{33e} \\ &+ 2(\bar{y}_1^* - \bar{Y}_1)(\bar{y}_2^* - \bar{Y}_2)\left(\frac{-1_1}{\bar{Y}_2^2}\right) + 2(\bar{y}_1^* - \bar{Y}_1)(\bar{x}^* - \bar{X})g_{10e} \\ &+ 2(\bar{y}_2^* - \bar{Y}_2)(\bar{x}^* - \bar{X})g_{20e} + 2(\bar{y}_1^* - \bar{Y}_1)(\hat{G} - G)g_{13e} \\ &+ 2(\bar{y}_2^* - \bar{Y}_2)(\hat{G} - G)g_{23e} + 2(\bar{x}^* - \bar{X})(\hat{G} - G)g_{03e}\} \\ &+ \ldots\end{aligned}$$

$$\left(R_{ge}^{*}-R\right)=\frac{e_1^*}{\bar{Y}_2}-\frac{Re_2^*}{\bar{Y}_2}+e_0^*G+e_3^*g_{3e}+\frac{1}{2!}\left\{\frac{2Re_2^{*2}}{\bar{Y}_2^2}\right.$$

$$+e_0^{*2}g_{00e}+e_3^{*2}g_{33e}-\frac{2e_1^*e_2^*}{\bar{Y}_2^2}$$

$$+2e_1^*e_0^*g_{10e}+2e_1^*e_3^*g_{13e}$$

$$\left.+2e_2^*e_0^*g_{20e}+2e_2^*e_3^*g_{23e}+2e_0^*e_3^*g_{03e}\right\}+\ldots \quad (2.9)$$

where $e_3^*=(\hat{G}-G),\ g(\bar{Y}_1,\bar{Y}_2,\bar{X},G)=R,$

$$g_{1e}=\left.\frac{\partial g(\bar{y}_1^*,\bar{y}_2^*,\bar{x}^*,\hat{G})}{\partial \bar{y}_1^*}\right]_T=\frac{1}{\bar{Y}_2},$$

$$g_{2e}=\left.\frac{\partial g(\bar{y}_1^*,\bar{y}_2^*,\bar{x}^*,\hat{G})}{\partial \bar{y}_2^*}\right]_T=-\frac{\bar{Y}_1}{\bar{Y}_2^2},$$

$$g_{0e}=\left.\frac{\partial g(\bar{y}_1^*,\bar{y}_2^*,\bar{x}^*,\hat{G})}{\partial \bar{x}^*}\right]_T,\quad g_{3e}=\left.\frac{\partial g(\bar{y}_1^*,\bar{y}_2^*,\bar{x}^*,\hat{G})}{\partial \hat{G}}\right]_T,$$

$$g_{11e}=\left.\frac{\partial^2 g(\bar{y}_1^*,\bar{y}_2^*,\bar{x}^*,\hat{G})}{\partial \bar{y}_1^{*2}}\right]_T=0,$$

$$g_{22e}=\left.\frac{\partial^2 g(\bar{y}_1^*,\bar{y}_2^*,\bar{x}^*,\hat{G})}{\partial \bar{y}_2^{*2}}\right]_T=\frac{2\bar{Y}_1}{\bar{Y}_2^3},$$

$$g_{00e}=\left.\frac{\partial^2 g(\bar{y}_1^*,\bar{y}_2^*,\bar{x}^*,\hat{G})}{\partial \bar{x}^{*2}}\right]_T,\ g_{33e}=\left.\frac{\partial^2 g(\bar{y}_1^*,\bar{y}_2^*,\bar{x}^*,\hat{G})}{\partial \hat{G}^2}\right]_T,$$

$$g_{10e}=\left.\frac{\partial^2 g(\bar{y}_1^*,\bar{y}_2^*,\bar{x}^*,\hat{G})}{\partial \bar{y}_1^*\partial \bar{x}^*}\right]_T,\ g_{20e}=\left.\frac{\partial^2 g(\bar{y}_1^*,\bar{y}_2^*,\bar{x}^*,\hat{G})}{\partial \bar{y}_2^*\partial \bar{x}^*}\right]_T,$$

$$g_{12e}=\left.\frac{\partial^2 g(\bar{y}_1^*,\bar{y}_2^*,\bar{x}^*,\hat{G})}{\partial \bar{y}_1^*\partial \bar{y}_2^*}\right]_T=-\frac{1}{\bar{Y}_2^2},$$

$$g_{13e}=\left.\frac{\partial^2 g\left(\bar{y}_1^*,\bar{y}_2^*,\bar{x}^*,\hat{G}\right)}{\partial \bar{y}_1^*\partial \hat{G}}\right]_T,\ g_{23e}=\left.\frac{\partial^2 g\left(\bar{y}_1^*,\bar{y}_2^*,\bar{x}^*,\hat{G}\right)}{\partial \bar{y}_2^*\partial \hat{G}}\right]_T,$$

$$g_{03e}=\left.\frac{\partial^2 g(\bar{y}_1^*,\bar{y}_2^*,\bar{x}^*,\hat{G})}{\partial \bar{x}^*\partial \hat{G}}\right]_T.$$

Taking expectation on both sides of (2.9), and ignoring terms in e_i's greater than two, the bias of $\hat{R}_{ge}^*$ to the first degree of approximation is of order $O\left(\frac{1}{n}\right)$; hence, for sufficiently large value of n, the bias is negligible.

The mean square error of $\hat{R}_{ge}^*$ given by $E(\hat{R}_{ge}^*-R)^2$ to the first degree of approximation is

$$MSE(\hat{R}_{ge}^*)=E\left\{\frac{e_1^*}{\bar{Y}_2}-\frac{Re_2^*}{\bar{Y}_2}+e_0^*G+e_3^*g_{3e}\right\}^2$$

$$=E\left\{\frac{e_1^{*2}}{\bar{Y}_2^2}+\frac{R^2e_2^{*2}}{\bar{Y}_2^2}-\frac{2Re_1^*e_2^*}{\bar{Y}_2^2}+e_0^{*2}G^2+e_3^{*2}g_{3e}^2+\frac{2Ge_0^*e_1^*}{\bar{Y}_2}\right.$$

$$\left.+2Ge_0^*e_3^*g_{3e}+\frac{2e_1^*e_3^*}{\bar{Y}_2}g_{3e}-\frac{2RGe_0^*e_2^*}{\bar{Y}_2}-\frac{2Re_2^*e_3^*}{\bar{Y}_2}g_{3e}\right\}$$

For $g_{3e}=0,\ MSE(\hat{R}_{ge}^*)$ becomes

$$MSE(\hat{R}_{ge}^*)=E\left\{\frac{e_1^*}{\bar{Y}_2}-\frac{Re_2^*}{\bar{Y}_2}+e_0^*G\right\}^2$$

$$=E\left\{\frac{e_1^{*2}}{\bar{Y}_2^2}+\frac{R^2e_2^{*2}}{\bar{Y}_2^2}-\frac{2Re_1^*e_2^*}{\bar{Y}_2^2}+e_0^{*2}G^2+\frac{2Ge_0^*e_1^*}{\bar{Y}_2}-\frac{2RGe_0^*e_2^*}{\bar{Y}_2}\right\}$$

$$=MSE(\hat{R}^*)$$

$$+\frac{\left[-\frac{1}{\bar{Y}_2}\left\{\frac{(1-f)}{n}\{S_{y_1x}-RS_{y_2x}\}+\frac{(k-1)W_2}{n}\{S_{y_1x_{(2)}}-RS_{y_2x_{(2)}}\}\right\}\right]^2}{\left[\frac{(1-f)}{n}S_x^2+\frac{(k-1)W_2}{n}S_{x_{(2)}}^2\right]}$$

$$-2\frac{\left[\frac{1}{\bar{Y}_2}\left\{\frac{(1-f)}{n}\{S_{y_1x}-RS_{y_2x}\}+\frac{(k-1)W_2}{n}\{S_{y_1x_{(2)}}-RS_{y_2x_{(2)}}\}\right\}\right]^2}{\left[\frac{(1-f)}{n}S_x^2+\frac{(k-1)W_2}{n}S_{x_{(2)}}^2\right]}$$

$$=MSE(\hat{R}^*)$$

$$-\frac{\left[\frac{1}{\bar{Y}_2}\left\{\frac{(1-f)}{n}\{S_{y_1x}-RS_{y_2x}\}+\frac{(k-1)W_2}{n}\{S_{y_1x_{(2)}}-RS_{y_2x_{(2)}}\}\right\}\right]^2}{\left[\frac{(1-f)}{n}S_x^2+\frac{(k-1)W_2}{n}S_{x_{(2)}}^2\right]} \quad (2.10)$$

which is equal to $MSE(\hat{R}_g^*)_{\min}$ in (2.6) if $g_{3e}=0$. Thus, the estimator $\hat{R}_{ge}^*=g(\bar{y}_1^*,\bar{y}_2^*,\bar{x}^*,\hat{G})$ attains the minimum mean square error of $\hat{R}_g^*$ in (2.6) if

$\hat{R}^*_{ge} = g(\bar{y}^*_1, \bar{y}^*_2, \bar{x}^*, \hat{G})$ is based on estimated optimum $\hat{G}$ satisfying at the point $T = (\bar{Y}_1, \bar{Y}_2, \bar{X}, G)$, the following conditions

$$\left.\begin{aligned} &g(\bar{Y}_1, \bar{Y}_2, \bar{X}, G) = R \\ &g_{1e} = \left.\frac{\partial g(\bar{y}^*_1, \bar{y}^*_2, \bar{x}^*, \hat{G})}{\partial \bar{y}^*_1}\right]_T = \frac{1}{\bar{Y}_2} \\ &g_{1e} = \left.\frac{\partial g(\bar{y}^*_1, \bar{y}^*_2, \bar{x}^*, \hat{G})}{\partial \bar{y}^*_1}\right]_T = \frac{1}{\bar{Y}_2} \\ &g_{0e} = \left.\frac{\partial g(\bar{y}^*_1, \bar{y}^*_2, \bar{x}^*, \hat{G})}{\partial \bar{x}^*}\right]_T = G \\ &g_{3e} = \left.\frac{\partial g(\bar{y}^*_1, \bar{y}^*_2 \bar{x}^*, \hat{G})}{\partial \hat{G}}\right]_T = 0 \end{aligned}\right\} \quad (2.11)$$

3. SOME PARTICULAR ESTIMATORS DEPENDING ON ESTIMATED OPTIMUM

(a) For the estimator $\hat{R}^*_1 = \dfrac{\bar{y}^*_1(\bar{X}+L)}{\bar{y}^*_2(\bar{x}^*+L)}$ on lines of Khare and Srivastava (1997), we have

$$g_0 = \left.\frac{\partial\{\bar{y}^*_1(\bar{X}+L)/\bar{y}^*_2(\bar{x}^*+L)\}}{\partial \bar{x}^*}\right]_{(\bar{Y}_1,\bar{Y}_2,\bar{X})} = -\frac{R}{(\bar{X}+L)}$$

which when substituted in (2.4), we get from the general expression for mean square error of $\hat{R}^*_g$ to be

$$MSE(\hat{R}^*_1) = MSE(\hat{R}^*) + \left(\frac{-R}{\bar{X}+L}\right)^2 \times \left\{\frac{(1-f)}{n}S_x^2 + \frac{(k-1)W_2}{n}S^2_{x(2)}\right\} - \frac{2}{\bar{Y}_2}\left(\frac{R}{\bar{X}+L}\right)\times\left[\frac{(1-f)}{n}\{S_{y_1x} - RS_{y_2x}\} + \frac{(k-1)N_2}{Nn}\{S_{y_1x(2)} - RS_{y_2x(2)}\}\right]$$

which is the same result as on lines of Khare and Srivastava (1997) as a special case of the general result in (2.4) of $\hat{R}^*_g$ for $g_0 = -\dfrac{R}{(\bar{X}+L)}$. Also, for the optimum value in (2.5) $g^*_0 = -\dfrac{R}{(\bar{X}+L)}$

$$= \frac{\left[-\frac{1}{\bar{Y}_2}\left\{\frac{(1-f)}{n}\{S_{y_1x} - RS_{y_2x}\} + \frac{(k-1)W_2}{n}\{S_{y_1x_{(2)}} - RS_{y_2x_{(2)}}\}\right\}\right]}{\left\{\frac{(1-f)}{n}S_x^2 + \frac{(k-1)W_2}{n}S^2_{x_{(2)}}\right\}}$$

$$= G \quad (3.1)$$

gives the optimum value of L to be

$$L_{opt} = \left[\frac{\left\{\frac{(1-f)}{n}S_x^2 + \frac{(k-1)W_2}{n}S^2_{x_{(2)}}\right\}R}{-\frac{1}{\bar{Y}_2}\left\{\frac{(1-f)}{n}\{S_{y_1x} - RS_{y_2x}\} + \frac{(k-1)W_2}{n}\{S_{y_1x_{(2)}} - RS_{y_2x_{(2)}}\}\right\}} - 1\right]$$

$$= -\frac{R}{G} - \bar{X} \quad (3.2)$$

and the minimum mean square error of $\hat{R}^*_1$ as a special case of the general $MSE(\hat{R}^*_g)_{\min}$ in (2.6) to be

$$MSE(\hat{R}^*_1) = MSE(\hat{R}^*) - \frac{\left[-\frac{1}{\bar{Y}_2}\left\{\frac{(1-f)}{n}\{S_{y_1x} - RS_{y_2x}\} + \frac{(k-1)W_2}{n}\{S_{y_1x_{(2)}} - RS_{y_2x_{(2)}}\}\right\}\right]^2}{\left\{\frac{(1-f)}{n}S_x^2 + \frac{(k-1)W_2}{n}S^2_{x_{(2)}}\right\}}$$

as on lines of Khare and Srivastava (1997). Considering the estimated optimum value $\hat{L}_{opt} = -\dfrac{\hat{R}^*}{\hat{G}} - \bar{X} = -\dfrac{\bar{y}^*_1}{\bar{y}^*_2\hat{G}} - \bar{X}$, we get the estimator depending on estimated optimum value $\hat{L}_{opt}$ to be

$$\hat{R}^*_{1e} = \frac{\bar{y}^*_1(\bar{X}+\hat{L}_{opt})}{\bar{y}^*_2(\bar{x}^*+\hat{L}_{opt})}$$

$$= \hat{R}^*\left(\bar{X} - \frac{\hat{R}^*}{\hat{G}} - \bar{X}\right)\Big/\left(\bar{x}^* - \frac{\hat{R}^*}{\hat{G}} - \bar{X}\right)$$

$$= \hat{R}^*\Big/\left\{1 - \frac{(\bar{x}^* - \bar{X})\hat{G}}{\hat{R}^*}\right\},$$

$$= \left(\frac{\bar{y}^*_1}{\bar{y}^*_2}\right)^2\Big/\left\{\frac{\bar{y}^*_1}{\bar{y}^*_2} - (\bar{x}^* - \bar{X})\hat{G}\right\}$$

satisfying the conditions of $\hat{R}^*_{ge} = g(\bar{y}^*_1, \bar{y}^*_2, \bar{x}^*, \hat{G})$ in (2.11), since

$\hat{R}^*_{1e}$ at $(\bar{Y}_1, \bar{Y}_2, \bar{X}, G)$

$$= \left(\frac{\bar{Y}_1}{\bar{Y}_2}\right)^2\Big/\left\{\frac{\bar{Y}_1}{\bar{Y}_2} - (\bar{X} - \bar{X})G\right\} = \frac{\bar{Y}_1}{\bar{Y}_2} = R$$

$$g_{1e} = \frac{\partial \hat{R}_{1e}^*}{\partial \overline{y}_1^*}\Bigg]_T = \left. \frac{\left[\left\{\left(\frac{\overline{y}_1^*}{\overline{y}_2^*}\right) - (\overline{x}^* - \overline{X})\hat{G}\right\}\frac{2\overline{y}_1^*}{\overline{y}_2^{*2}} - \left(\frac{\overline{y}_1^{*2}}{\overline{y}_2^{*3}}\right)\right]}{\left\{\left(\frac{\overline{y}_1^*}{\overline{y}_2^*}\right) - (\overline{x}^* - \overline{X})\hat{G}\right\}^2}\right]_T$$

$$= \frac{1}{\overline{Y}_2}$$

$$g_{2e} = \frac{\partial \hat{R}_{2e}^*}{\partial \overline{y}_2^*}\Bigg]_T$$

$$= \left. \frac{\left[\left\{\left(\frac{\overline{y}_1^*}{\overline{y}_2^*}\right) - (\overline{x}^* - \overline{X})\hat{G}\right\}\left(-\frac{2\overline{y}_1^{*2}}{\overline{y}_2^{*3}}\right) + \left(\frac{\overline{y}_1^{*3}}{\overline{y}_2^{*4}}\right)\right]}{\left\{\left(\frac{\overline{y}_1^*}{\overline{y}_2^*}\right) - (\overline{x}^* - \overline{X})\hat{G}\right\}^2}\right]_T = -\frac{\overline{Y}_1}{\overline{Y}_2^2}$$

$$g_{0e} = \frac{\partial \hat{R}_{1e}^*}{\partial \overline{x}^*}\Bigg]_T = \left. \frac{\left[\left\{\left(\frac{\overline{y}_1^*}{\overline{y}_2^*}\right) - (\overline{x}^* - \overline{X})\hat{G}\right\}(0) + \left(\frac{\overline{y}_1^{*2}}{\overline{y}_2^{*2}}\right)\hat{G}\right]}{\left\{\left(\frac{\overline{y}_1^*}{\overline{y}_2^*}\right) - (\overline{x}^* - \overline{X})\hat{G}\right\}^2}\right]_T$$

$$= G$$

$$g_{3e} = \frac{\partial \hat{R}_{1e}^*}{\partial \hat{G}}\Bigg]_T$$

$$= \left. \frac{\left[\left\{\left(\frac{\overline{y}_1^*}{\overline{y}_2^*}\right) - (\overline{x}^* - \overline{X})\hat{G}\right\}(0) + \left(\frac{\overline{y}_1^{*2}}{\overline{y}_2^{*2}}\right)(\overline{x}^* - \overline{X})\right]}{\left\{\left(\frac{\overline{y}_1^*}{\overline{y}_2^*}\right) - (\overline{x}^* - \overline{X})\hat{G}\right\}^2}\right]_T = 0;$$

hence, the estimator $\hat{R}_{1e}^* = \left(\frac{\overline{y}_1^*}{\overline{y}_2^*}\right)^2 \Big/ \left\{\frac{\overline{y}_1^*}{\overline{y}_2^*} - (\overline{x}^* - \overline{X})\hat{G}\right\}$ depending on estimated optimum value attains the minimum mean square error in (2.10) given by

$$MSE(\hat{R}_{1e}^*)_{\min} = MSE(\hat{R}^*)$$

$$-\frac{\left[\frac{1}{\overline{Y}_2}\left\{\frac{(1-f)}{n}\{S_{y_1x} - RS_{y_2x}\} + \frac{(k-1)W_2}{n}\{S_{y_1x_{(2)}} - RS_{y_2x_{(2)}}\}\right\}\right]^2}{\left\{\frac{(1-f)}{n}S_x^2 + \frac{(k-1)W_2}{n}S_{x_{(2)}}^2\right\}}.$$

(b) For the estimator $\hat{R}_4^* = \frac{\overline{y}_1^*}{\overline{y}_2^*}e^{\alpha\left(\frac{\overline{x}^*}{\overline{X}}-1\right)}$, we have

$$g_0 = \frac{\partial \hat{R}^* e^{\alpha\left(\frac{\overline{x}^*}{\overline{X}}-1\right)}}{\partial \overline{x}^*}\Bigg]_{(\overline{Y}_1,\overline{Y}_2,\overline{X})} = Re^{\alpha\left(\frac{\overline{X}}{\overline{X}}-1\right)}\alpha\left(\frac{1}{\overline{X}}\right) = \alpha\frac{R}{\overline{X}}$$

which when substituted in (2.4), we get directly the mean square error of the estimator $\hat{R}_4^* = \frac{\overline{y}_1^*}{\overline{y}_2^*}e^{\alpha\left(\frac{\overline{x}^*}{\overline{X}}-1\right)}$ to be

$$MSE(\hat{R}_4^*)$$

$$= MSE(\hat{R}^*) + \alpha^2\frac{R^2}{\overline{X}^2}\left\{\frac{(1-f)}{n}S_x^2 + \frac{(k-1)W_2}{n}S_{x(2)}^2\right\}$$

$$+2\frac{\alpha}{\overline{Y}_2}\frac{R}{\overline{X}}\left\{\frac{(1-f)}{n}\{S_{y_1x} - RS_{y_2x}\}\right.$$

$$\left. + \frac{(k-1)W_2}{n}\{S_{y_1x(2)} - RS_{y_2x(2)}\}\right\}.$$

Minimizing value of g_0 is

$$g_0^* = \frac{\alpha R}{\overline{X}}$$

$$= \frac{\left[-\frac{1}{\overline{Y}_2}\left\{\frac{(1-f)}{n}\{S_{y_1x} - RS_{y_2x}\} + \frac{(k-1)W_2}{n}\{S_{y_1x_{(2)}} - RS_{y_2x_{(2)}}\}\right\}\right]}{\left\{\frac{(1-f)}{n}S_x^2 + \frac{(k-1)W_2}{n}S_{x_{(2)}}^2\right\}}$$

$$= G$$

giving $\alpha = \frac{G\overline{X}}{R}$ which when substituted in (2.6), gives the minimum mean square error of $\hat{R}_4^*$ to be

$$MSE(\hat{R}_4^*) = MSE(\hat{R}^*)$$

$$-\frac{\left[\frac{1}{\overline{Y}_2}\left\{\frac{(1-f)}{n}\{S_{y_1x} - RS_{y_2x}\} + \frac{(k-1)W_2}{n}\{S_{y_1x_{(2)}} - RS_{y_2x_{(2)}}\}\right\}\right]^2}{\left\{\frac{(1-f)}{n}S_x^2 + \frac{(k-1)W_2}{n}S_{x_{(2)}}^2\right\}}$$

Now, considering the estimated optimum value $\hat{g}_0^* = \frac{\hat{\alpha}\hat{R}}{\overline{x}^*} = \left(\frac{\overline{y}_1^*\hat{\alpha}}{\overline{y}_2^*\overline{x}^*}\right) = \hat{G}$ giving $\hat{\alpha} = \frac{\hat{G}\,\overline{y}_2^*\,\overline{x}^*}{\overline{y}_1^*}$ which gives the estimator depending on estimated optimum value to be $\hat{R}^*_{4e} = \frac{\overline{y}_1^*}{\overline{y}_2^*} e^{\frac{\hat{G}\overline{y}_2^*\overline{x}^*}{\overline{y}_1^*}\left(\frac{\overline{x}^*}{\overline{X}}-1\right)}$ satisfying the conditions

(i) $$\left.\frac{\overline{y}_1^*}{\overline{y}_2^*} e^{\frac{\hat{G}\overline{y}_2^*\overline{x}^*}{\overline{y}_1^*}\left(\frac{\overline{x}^*}{\overline{X}}-1\right)}\right]_{(\overline{Y}_1,\overline{Y}_2,\overline{X},G)} = \frac{\overline{Y}_1}{\overline{Y}_2} = R$$

(ii)
$$g_{1e} = e^{\frac{\hat{G}\overline{y}_2^*\overline{x}^*}{\overline{y}_1^*}\left(\frac{\overline{x}^*}{\overline{X}}-1\right)}\left.\left\{-\frac{\overline{y}_1^*}{\overline{y}_2^*}\frac{\hat{G}\overline{y}_2^*\overline{x}^*}{\overline{y}_1^{*2}}\left(\frac{\overline{x}^*}{\overline{X}}-1\right)+\frac{1}{\overline{y}_2^*}\right\}\right]_{(\overline{Y}_1,\overline{Y}_2,\overline{X},G)}$$
$$= \frac{1}{\overline{Y}_2}$$

(iii)
$$g_{2e} = e^{\frac{\hat{G}\overline{y}_2^*\overline{x}^*}{\overline{y}_1^*}\left(\frac{\overline{x}^*}{\overline{X}}-1\right)}\left.\left\{\frac{\overline{y}_1^*}{\overline{y}_2^*}\frac{\hat{G}\overline{x}^*}{\overline{y}_1^*}\left(\frac{\overline{x}^*}{\overline{X}}-1\right)-\frac{\overline{y}_1^*}{\overline{y}_2^{*2}}\right\}\right]_{(\overline{Y}_1,\overline{Y}_2,\overline{X},G)}$$
$$= -\frac{\overline{Y}_1}{\overline{Y}_2^2}$$

(iv) $$g_{0e} = \frac{\overline{y}_1^*}{\overline{y}_2^*} e^{\frac{\hat{G}\overline{y}_2^*\overline{x}^*}{\overline{y}_1^*}\left(\frac{\overline{x}^*}{\overline{X}}-1\right)}\left.\left\{\hat{G}\frac{\overline{y}_2^*}{\overline{y}_1^*}\left(2\frac{\overline{x}^*}{\overline{X}}-1\right)\right\}\right]_{(\overline{Y}_1,\overline{Y}_2,\overline{X},G)}$$
$$= G$$

and

(v) $$g_{3e} = \frac{\overline{y}_1^*}{\overline{y}_2^*} e^{\frac{\hat{G}\overline{y}_2^*\overline{x}^*}{\overline{y}_1^*}\left(\frac{\overline{x}^*}{\overline{X}}-1\right)}\left.\left\{\frac{\overline{y}_2^*\overline{x}^*}{\overline{y}_1^*}\left(\frac{\overline{x}^*}{\overline{X}}-1\right)\right\}\right]_{(\overline{Y}_1,\overline{Y}_2,\overline{X},G)}$$
$$= 0$$

of (2.11); hence, the estimator $\hat{R}^*_{4e} = \frac{\overline{y}_1^*}{\overline{y}_2^*} e^{\frac{\hat{G}\overline{y}_2^*\overline{x}^*}{\overline{y}_1^*}\left(\frac{\overline{x}^*}{\overline{X}}-1\right)}$ with estimated optimum $\hat{G}$ in (2.7) attains the minimum mean square error in (2.10) given by

$$MSE(\hat{R}^*_{4e})_{\min} = MSE(\hat{R}^*)$$
$$-\frac{\left[\frac{1}{\overline{Y}_2}\left\{\frac{(1-f)}{n}\{S_{y_1x}-RS_{y_2x}\}+\frac{(k-1)W_2}{n}\{S_{y_1x(2)}-RS_{y_2x(2)}\}\right\}\right]^2}{\left\{\frac{(1-f)}{n}S_x^2+\frac{(k-1)W_2}{n}S_{x(2)}^2\right\}}.$$

(c) For the estimator $\hat{R}^*_8 = \frac{\overline{y}_1^* + \alpha(\overline{x}^* - \overline{X})}{\overline{y}_2^*}$, we have $g_0 = \left.\frac{\partial\{\overline{y}_1^* + \alpha(\overline{x}^* - \overline{X})/\overline{y}_2^*\}}{\partial\overline{x}^*}\right]_{(\overline{Y}_1,\overline{Y}_2,\overline{X})} = \frac{\alpha}{\overline{Y}_2}$ which when substituted in (2.4), we get from the general expression for mean square error of $\hat{R}^*_g$ to be

$$MSE(\hat{R}^*_8) = MSE(\hat{R}^*)$$
$$+\left(\frac{\alpha}{\overline{Y}_2}\right)^2\left\{\frac{(1-f)}{n}S_x^2+\frac{(k-1)W_2}{n}S_{x(2)}^2\right\}$$
$$+\frac{2}{\overline{Y}_2}\left(\frac{\alpha}{\overline{Y}_2}\right)\left[\frac{(1-f)}{n}\{S_{y_1x}-RS_{y_2x}\}\right.$$
$$\left.+\frac{(k-1)N_2}{Nn}\{S_{y_1x(2)}-RS_{y_2x(2)}\}\right]$$

which is the same result as the mean square error of $\hat{R}^*_8$. Also, for the optimum value in (2.5)

$$g_0^* = \left(\frac{\alpha}{\overline{Y}_2}\right)$$
$$= \frac{\left[-\frac{1}{\overline{Y}_2}\left\{\frac{(1-f)}{n}\{S_{y_1x}-RS_{y_2x}\}+\frac{(k-1)W_2}{n}\{S_{y_1x(2)}-RS_{y_2x(2)}\}\right\}\right]}{\left\{\frac{(1-f)}{n}S_x^2+\frac{(k-1)W_2}{n}S_{x(2)}^2\right\}}$$
$$= G$$

gives the same optimum value of α to be

$$\alpha_{opt}$$
$$= \frac{\left[-\left\{\frac{(1-f)}{n}\{S_{y_1x}-RS_{y_2x}\}+\frac{(k-1)W_2}{n}\{S_{y_1x(2)}-RS_{y_2x(2)}\}\right\}\right]}{\left\{\frac{(1-f)}{n}S_x^2+\frac{(k-1)W_2}{n}S_{x(2)}^2\right\}}$$
$$= G\overline{Y}_2$$

and the same minimum mean square error of $\hat{R}^*_8$ as a special case of the general $MSE(\hat{R}^*_g)_{\min}$ in (2.6) to be

$$MSE(\hat{R}^*_8) = MSE(\hat{R}^*)$$
$$-\frac{\left[\frac{1}{\overline{Y}_2}\left\{\frac{(1-f)}{n}\{S_{y_1x}-RS_{y_2x}\}+\frac{(k-1)W_2}{n}\{S_{y_1x(2)}-RS_{y_2x(2)}\}\right\}\right]^2}{\left\{\frac{(1-f)}{n}S_x^2+\frac{(k-1)W_2}{n}S_{x(2)}^2\right\}}.$$

Considering the estimated optimum value $\hat{\alpha}_{opt} = \hat{G}\overline{y}_2^*$, we get the estimator depending on estimated optimum value $\hat{\alpha}_{opt}$ to be

$$\hat{R}_{8e}^* = \frac{\overline{y}_1^* + \hat{\alpha}_{opt}(\overline{x}^* - \overline{X})}{\overline{y}_2^*}$$

$$= \frac{\overline{y}_1^* + \hat{G}\overline{y}_2^*\left(\overline{x}^* - \overline{X}\right)}{\overline{y}_2^*}$$

$$= \frac{\overline{y}_1^*}{\overline{y}_2^*} + \hat{G}(\overline{x}^* - \overline{X})$$

satisfying the conditions of $\hat{R}_{ge}^* = g(\overline{y}_1^*, \overline{y}_2^*, \overline{x}^*, \hat{G})$ in (2.11), since

$$\hat{R}_{8e}^* \text{ at } (\overline{Y}_1, \overline{Y}_2, \overline{X}, G) = R + G(\overline{X} - \overline{X}) = R$$

$$g_{1e} = \left.\frac{\partial \hat{R}_{8e}^*}{\partial \overline{y}_1^*}\right]_T = \left.\left(\frac{1}{\overline{y}_2^*}\right)\right]_T = \frac{1}{\overline{Y}_2}$$

$$g_{2e} = \left.\frac{\partial \hat{R}_{8e}^*}{\partial \overline{y}_2^*}\right]_T = \left.\left(\frac{-\overline{y}_1^*}{\overline{y}_2^{*2}}\right)\right]_T = -\frac{\overline{Y}_1}{\overline{Y}_2^2}$$

$$g_{0e} = \left.\frac{\partial \hat{R}_{8e}^*}{\partial \overline{x}^*}\right]_T = \hat{G}\Big]_T = G$$

$$g_{3e} = \left.\frac{\partial \hat{R}_{8e}^*}{\partial \hat{G}}\right]_T = 0;$$

hence, the estimator $\hat{R}_8^* = \dfrac{\overline{y}_1^* + \alpha(\overline{x}^* - \overline{X})}{\overline{y}_2^*}$ depending on estimated optimum value attains the minimum mean square error in (2.10) given by

$$MSE(\hat{R}_{8e}^*)_{\min} = MSE(\hat{R}^*)$$

$$-\frac{\left[\dfrac{1}{\overline{Y}_2}\left\{\dfrac{(1-f)}{n}\{S_{y_1x} - RS_{y_2x}\} + \dfrac{(k-1)W_2}{n}\{S_{y_1x_{(2)}} - RS_{y_2x_{(2)}}\}\right\}\right]^2}{\left\{\dfrac{(1-f)}{n}S_x^2 + \dfrac{(k-1)W_2}{n}S_{x_{(2)}}^2\right\}}.$$

(d) For the conventional ratio estimator $\hat{R}_0^* = \dfrac{\overline{y}_1^*}{\overline{y}_2^*}\dfrac{\overline{X}}{\overline{x}^*}$ on the lines of Hansen and Hurwitz (1946), we have $g_0 = -\dfrac{R}{\overline{X}}$ which when substituted in (2.4), we get the mean square error of $\hat{R}_0^*$ as a special case of $\hat{R}_g^*$ to be

$$MSE(\hat{R}_0^*) = MSE(\hat{R}^*) + \left\{\frac{(1-f)}{n}S_x^2 + \frac{(k-1)W_2}{n}S_{x_{(2)}}^2\right\} \times$$

$$\frac{R^2}{\overline{X}^2} - 2\left\{\frac{(1-f)}{n}\{S_{y_1x} - RS_{y_2x}\}\right.$$

$$\left. + \frac{(k-1)N_2}{Nn}\{S_{y_1x_{(2)}} - RS_{y_2x_{(2)}}\}\right\}\frac{R}{\overline{Y}_2\overline{X}}$$

It may be mentioned here that $\hat{R}_0^*$ cannot attain the minimum mean square error since $\hat{R}_0^*$ does not satisfy the conditions of G or $\hat{G}$.

(e) All the results of the remaining estimators $\hat{R}_2^*$, $\hat{R}_3^*$ and $\hat{R}_5^*$ to $\hat{R}_7^*$ may be easily obtained as special cases of the general results given in (2.4), (2.5), (2.6) and (2.10) as are found in (a), (b) and (c).

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